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Small-threshold behaviour of two-loop self-energy diagrams:  
two-particle thresholds <sup>1</sup>

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**Abstract**

The behaviour of two-loop two-point diagrams at non-zero thresholds corresponding to two-particle cuts is analyzed. The masses involved in a cut and the external momentum are assumed to be small as compared to some of the other masses of the diagram. By employing general formulae of asymptotic expansions of Feynman diagrams in momenta and masses, we construct an algorithm to derive analytic approximations to the diagrams. In such a way, we calculate several first coefficients of the expansion. Since no conditions on relative values of the small masses and the external momentum are imposed, the threshold irregularities are described analytically. Numerical examples, using diagrams occurring in the Standard Model, illustrate the convergence of the expansion below the first large threshold.

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## 1. Introduction

In recent years many precision calculations have been performed in the Standard Model (SM), which are required to match the impressive experimental accuracy. A good overview of the present status has been recently given [1]. It is to be expected that even more refined evaluations of loop corrections in the SM and extensions thereof will be needed in the future. Since one-loop corrections in the SM are well established the focus will be on two-loop studies. As physics examples one should mention the quantity  $\Delta r$  occurring in the Fermi constant  $G_F$ , small angle Bhabha scattering and the  $Z \rightarrow b\bar{b}$  decay rate.

Such calculations are cumbersome in many respects e.g. through the number of diagrams, the involved tensor structures in diagrams and the complexity of basic building blocks i.e. scalar two-loop diagrams<sup>1</sup>.

It is the purpose of this paper to contribute to the techniques for calculation of the scalar two-loop diagrams. When all masses in the propagators are non-vanishing there exist arguments [3] that they cannot be expressed anymore in terms of known functions like polylogarithms. As an example, we can mention the simplest case of a two-loop scalar diagram i.e. a self energy with three massive propagators, the so-called “sunset” (alias “sunrise”) diagram, which can be expressed in terms of Lauricella functions [4] or one-fold integrals [5, 6], but no simpler results are available so far.

In such a situation two main strategies can be chosen. One is a numerical approach, another an approximative analytical method. Most progress has been made for self-energy diagrams. Obviously they are the simplest ones, but they are physics-wise also very relevant.

As to the numerical strategy one still has a number of options. There exist a two-dimensional integral representation [7] (see also in [8]) originally introduced for the master diagram (Fig. 1a), but later on extended to all (also divergent) two-loop self-energies [9]. Another way is the use of dispersion integrals [4, 10], whereas other methods rely on higher dimensional integration techniques [11, 5].

The analytical approach uses explicit formulae for the asymptotic expansion of Feynman diagrams in momenta and masses and is based on general theorems on asymptotic expansions [12] (see also [13] for review). Some recent physical examples of application of these formulae can be found e.g. in [14]. For two-loop self-energy diagrams with general masses, expansions in different regions were systematically examined in refs. [15, 16, 17]<sup>2</sup>.

When a few terms of the series give an adequate description this approach will be faster than the purely numerical integration method. Moreover, analytic expressions are much more convenient to deal with when the values of some masses are not fixed (e.g. the Higgs mass) or have rather wide error bars (e.g. the top quark mass).

The first results of the analytical approach have been obtained outside the particle thresholds: a small momentum expansion below the lowest threshold [15] and a large mo-

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<sup>1</sup>The tensor decomposition of two-loop self-energy diagrams is described in ref. [2].

<sup>2</sup>The small momentum expansion in the three-point case was studied in ref. [18] where also the techniques of conformal mapping and Padé approximations were employed to get numerical results beyond the threshold(s).

momentum expansion above the highest threshold [16]<sup>3</sup>. The region between the thresholds poses a problem since both small and large momentum expansions do not describe the behaviour between the lowest and the highest physical thresholds. In a previous paper [17] it was shown that in the special case of a zero mass threshold the small momentum expansion could be extended to the lowest non-vanishing threshold. In this case the expansion coefficients involve the zero-threshold cut which appears as powers of  $\ln(-k^2)$  (where  $k$  is the external momentum).

It is the purpose of the present paper to consider cases when one (or two) two-particle threshold(s) is (are) small with respect to the other thresholds, but not anymore zero. By using asymptotic expansions in the large mass limit it becomes possible to find a series converging above the small two-particle threshold. The expansion coefficients now contain the two-particle cut(s) associated with the small threshold(s) and therefore the non-regular behaviour around the threshold(s) will be exactly described. Thus the analytic approach now is substantially extended into regions, which were hitherto inaccessible due to the inapplicability of the used series. From numerical examples it can be seen that the convergence holds beyond the small particle threshold(s). We note that there are no exact results (except for the zero-threshold limit) available for the cases we are going to describe here.

The actual outline of the paper is as follows. In section 2 different small-threshold configurations are considered and subgraphs contributing to the asymptotic expansions of the corresponding diagrams are described. In section 3 we present some analytical results for the lowest terms of the expansions. Section 4 contains a numerical comparison between the direct numerical integration of the Feynman diagram and the results of our analytic approximations. Conclusions and future prospects are given in section 4. In Appendix A we collect some relevant information on one-loop two-point integrals with different masses and present some formulae to handle the numerators. In Appendix B we present a result for one of the next-to-leading terms of the expansion.

## 2. Constructing the expansion

In Fig. 1a,b two different topologies of two-loop self-energy diagrams are shown. The dimensionally-regularized scalar Feynman integrals corresponding to these diagrams can be written as<sup>4</sup>

$$J(\{\nu_i\}; \{m_i\}; k) = \int \int \frac{d^n p \, d^n q}{D_1^{\nu_1} D_2^{\nu_2} D_3^{\nu_3} D_4^{\nu_4} D_5^{\nu_5}}, \quad (1)$$

where  $n = 4 - 2\varepsilon$  is the space-time dimension [21], and  $(D_i)^{\nu_i} \equiv (p_i^2 - m_i^2 + i0)^{\nu_i}$  are the powers of the denominators (scalar propagators) corresponding to the internal lines of the diagrams in Fig. 1a,b. The momenta  $p_i$  flowing through the internal lines are constructed, in an obvious way, from the external momentum  $k$  and the loop momenta  $p$  and  $q$ .

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<sup>3</sup>For some special cases (when some of the internal lines are massless), the expansions and exact results were presented in [19, 20].

<sup>4</sup>In many cases, we adopt the notation used in the previous papers [15, 16, 17].

We are mainly interested in the cases when all  $\nu$ 's are integer. As mentioned in refs. [15, 16, 17], in this case it is enough to consider the “master” diagram (Fig. 1a) only, because (i) the diagrams with four and three internal lines can be obtained from Fig. 1a by shrinking (reducing to points) some of the internal lines (this corresponds to putting the corresponding  $\nu$ 's equal to zero) and (ii) the diagram in Fig. 1b can be reduced to the diagrams with four denominators<sup>5</sup>. Thus, in what follows we shall imply that the integral (1) corresponds to the diagram in Fig. 1a.

In general, the diagram in Fig. 1a has two two-particle thresholds (at  $k^2 = (m_1 + m_4)^2$  and  $k^2 = (m_2 + m_5)^2$ ) and two three-particle thresholds (at  $k^2 = (m_1 + m_3 + m_5)^2$  and  $k^2 = (m_2 + m_3 + m_4)^2$ ). If all these physical thresholds correspond to non-zero (positive) values of  $k^2$ , the small momentum expansion is nothing but a Taylor expansion in  $k^2$ . In [15] a general algorithm for calculating the coefficients of this expansion was developed. Otherwise, if we have one (or more) physical threshold(s) at  $k^2 = 0$ , the corresponding “zero-threshold” expansion (as  $n \rightarrow 4$ ) involves also the terms with  $\ln(-k^2)$  (and even  $\ln^2(-k^2)$  when two physical thresholds vanish). To calculate analytically the coefficients of zero-threshold expansion of the diagram of Fig. 1a, in [17] general theorems on asymptotic expansions of Feynman diagrams [12] were applied. In such a way, all different configurations of massless thresholds have been considered. In particular, it has been verified that the expansion converges up to the first non-zero threshold.

In this paper, we study the behaviour of two-loop self-energy diagrams which have small (but non-zero) physical thresholds. Namely, let us consider some of the masses corresponding to internal lines of Fig. 1a to be large, while the other masses as well as the external momentum are small (and of the same order). To distinguish between the small and the large masses, we shall denote the latter with capital letters,  $M_i$ . By analogy with ref. [17], four different small-threshold configurations exist, namely<sup>6</sup>:

Case 1: one small 2PT (e.g. masses  $m_2$  and  $m_5$  are small);

Case 2: two small 2PT's (the masses  $m_1, m_2, m_4, m_5$  are small);

Case 3: one small 3PT (e.g. masses  $m_2, m_3, m_4$  are small);

Case 4: one small 2PT and one small 3PT (e.g. the masses  $m_2, m_3, m_4, m_5$  are small).

In addition, two special subcases of case 1, when one more mass (not involved in the threshold) is small, should be considered separately:

Case 1a: case 1 with  $m_3$  being small;

Case 1b: case 1 with one of the masses involved in the second 2PT being small.

In the present paper we restrict ourselves to cases 1, 1a, 1b and 2 where only small *two*-particle thresholds are involved (see a brief discussion of the cases 3 and 4 in the last section).

Before employing general results on asymptotic expansions in momenta and masses [12], we need to introduce some notation. We shall need Taylor operators  $\mathcal{T}_k$  and  $\mathcal{T}_m$  expanding the denominators in the integrand (in the momentum or the mass, respectively)

<sup>5</sup>For the diagram in Fig. 1b,  $p_1 = p_4$ . If, in addition,  $m_1 = m_4$ , we get the integral with four propagators (with one the powers equal to  $\nu_1 + \nu_4$ ). If  $m_1 \neq m_4$  and both  $\nu_1$  and  $\nu_4$  are integer, one can use standard partial fractioning to reduce the diagram in Fig. 1b to a linear combination of the integrals with  $\nu_1$  or  $\nu_4$  equal to zero. All these integrals can be obtained from the diagram in Fig. 1a by shrinking one of the lines.

<sup>6</sup>2PT and 3PT mean two- and three-particle thresholds, respectively.

as

$$\mathcal{T}_k \frac{1}{[(k-p)^2 - m^2]^\nu} = \sum_{j=0}^{\infty} \frac{(\nu)_j}{j!} \frac{(2(kp) - k^2)^j}{[p^2 - m^2]^{\nu+j}}, \quad (2)$$

$$\mathcal{T}_m \frac{1}{[p^2 - m^2]^\nu} = \frac{1}{(p^2)^\nu} \sum_{j=0}^{\infty} \frac{(\nu)_j}{j!} \left( \frac{m^2}{p^2} \right)^j, \quad (3)$$

where

$$(\nu)_j \equiv \frac{\Gamma(\nu + j)}{\Gamma(\nu)} \quad (4)$$

is the Pochhammer symbol. The “combined” operators  $\mathcal{T}$  expanding in some momenta and masses at the same time can be constructed as products of the above operators (2) and (3). It is assumed that the Taylor operators are applied to the integrand before the loop integration is performed. After using eqs. (2)–(3), we should collect together all terms carrying the same *total* power of the small masses and momenta we expand in.

Let  $\Gamma$  be the original graph (corresponding in our case to Fig. 1a), subgraphs of  $\Gamma$  are denoted as  $\gamma$ , and the corresponding “reduced graph”  $\Gamma/\gamma$  is obtained from  $\Gamma$  by shrinking the subgraph  $\gamma$  to a point. Furthermore,  $J_\gamma$  is the dimensionally-regularized Feynman integral with the denominators corresponding to a graph  $\gamma$ . In particular,  $J_\Gamma$  corresponds to the integral (1) itself. Then, for our case, the general theorem yields

$$J_\Gamma \underset{k, m_i \rightarrow 0}{\sim} \sum_{\gamma} J_{\Gamma/\gamma} \circ \mathcal{T}_{k, m_i, q_i} J_\gamma, \quad (5)$$

where the sum goes over all the subgraphs  $\gamma$  which (i) contain all the lines with the large masses  $M_i$ , and (ii) are one-particle irreducible with respect to the light lines (with the masses  $m_i$ ). The Taylor operator  $\mathcal{T}_{k, m_i, q_i}$  (see eqs. (2)–(3)) expands the integrand of  $J_\gamma$  in small masses  $m_i$ , the external momentum  $k$  and the loop momenta  $q_i$  which are “external” for a given subgraph  $\gamma$ . The symbol “ $\circ$ ” means that the polynomial in  $q_i$ , which appears as a result of applying  $\mathcal{T}$  to  $J_\gamma$ , should be inserted in the numerator of the integrand of  $J_{\Gamma/\gamma}$ .

The definition of the class of subgraphs  $\gamma$  involved in the expansion (5) is exactly the same as in the zero-threshold expansion [17], just because zero masses are always small. A distinction is that some tadpole-like contributions (vanishing in zero-threshold limit) should be taken into account for small non-zero masses (see below).

Let us consider which subgraphs  $\gamma$  contribute to the sum (5) for the different small-threshold configurations. To indicate which lines are included in the subgraph, we shall use the numbering of lines given in Fig. 1a. For example, the “full” graph  $\Gamma \equiv \{12345\}$  includes all five lines,  $\{134\}$  corresponds to a subgraph where the lines 2 and 5 are “omitted”, etc. The subgraphs contributing to the expansion are shown in Fig. 2, where bold and narrow lines correspond to the propagators with large and small masses, respectively. Dotted lines indicate the lines omitted in the subgraphs  $\gamma$  (as compared with the “full” graph  $\Gamma$ ), i.e. they correspond to the reduced graphs  $\Gamma/\gamma$ .

Comparing Fig. 2 with the four first lines of Fig. 3 presented in [17] (zero-threshold expansion), one can see that Fig. 2 contains some additional subgraphs for the cases 1a

and 1b (namely, the second and the fourth subgraphs for each of these cases). These subgraphs did not contribute to the zero-threshold expansion since the corresponding reduced graphs  $\Gamma/\gamma$  yielded massless tadpole-like integrals and therefore vanished in dimensional regularization [21]. Here, we get tadpoles with small masses, which do *not* vanish and must be taken into account<sup>7</sup>.

*Case 1.* Two subgraphs contribute to the asymptotic expansion:

(i)  $\gamma = \Gamma \Rightarrow$

$$\int \int d^n p d^n q \frac{1}{[(p-q)^2 - M_3^2]^{\nu_3} [p^2 - M_4^2]^{\nu_4}} \times \mathcal{T}_{k,m_2,m_5} \frac{1}{[q^2 - m_5^2]^{\nu_5} [(k-p)^2 - M_1^2]^{\nu_1} [(k-q)^2 - m_2^2]^{\nu_2}}; \quad (6)$$

(ii)  $\gamma = \{134\} \Rightarrow$

$$\int \int d^n p d^n q \frac{1}{[(k-q)^2 - m_2^2]^{\nu_2} [q^2 - m_5^2]^{\nu_5}} \times \mathcal{T}_{k,q} \frac{1}{[(k-p)^2 - M_1^2]^{\nu_1} [(p-q)^2 - M_3^2]^{\nu_3} [p^2 - M_4^2]^{\nu_4}}. \quad (7)$$

*Case 1a.* Four subgraphs contribute to the asymptotic expansion:

(i)  $\gamma = \Gamma \Rightarrow$

$$\int \int d^n p d^n q \frac{1}{[p^2 - M_4^2]^{\nu_4}} \times \mathcal{T}_{k,m_2,m_3,m_5} \frac{1}{[(p-q)^2 - m_3^2]^{\nu_3} [q^2 - m_5^2]^{\nu_5} [(k-p)^2 - M_1^2]^{\nu_1} [(k-q)^2 - m_2^2]^{\nu_2}}; \quad (8)$$

(ii)  $\gamma = \{1245\} \Rightarrow$

$$\int \int d^n p d^n q \frac{1}{[p^2 - m_3^2]^{\nu_3}} \times \mathcal{T}_{k,p,m_2,m_5} \frac{1}{[(p+q)^2 - M_4^2]^{\nu_4} [q^2 - m_5^2]^{\nu_5} [(k-p-q)^2 - M_1^2]^{\nu_1} [(k-q)^2 - m_2^2]^{\nu_2}}; \quad (9)$$

(iii)  $\gamma = \{134\} \Rightarrow$

$$\int \int d^n p d^n q \frac{1}{[(k-q)^2 - m_2^2]^{\nu_2} [q^2 - m_5^2]^{\nu_5}} \times \mathcal{T}_{k,q,m_3} \frac{1}{[(k-p)^2 - M_1^2]^{\nu_1} [(p-q)^2 - m_3^2]^{\nu_3} [p^2 - M_4^2]^{\nu_4}}; \quad (10)$$

(iv)  $\gamma = \{14\} \Rightarrow$

$$\int \int d^n p d^n q \frac{1}{[p^2 - m_3^2]^{\nu_3} [q^2 - m_5^2]^{\nu_5} [(k-q)^2 - m_2^2]^{\nu_2}} \times \mathcal{T}_{k,p,q} \frac{1}{[(p+q)^2 - M_4^2]^{\nu_4} [(k-p-q)^2 - M_1^2]^{\nu_1}}; \quad (11)$$

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<sup>7</sup>Appearance of extra subgraphs for the cases 1a and 1b does not contradict the fact that all the cases 1, 1a and 1b correspond to the same small-threshold configuration (one small 2PT). We shall discuss details of this connection in section 3.

*Case 1b.* Four subgraphs contribute:

(i)  $\gamma = \Gamma \Rightarrow$

$$\int \int d^n p d^n q \frac{1}{[(p-q)^2 - M_3^2]^{\nu_3}} \times \mathcal{T}_{k,m_2,m_4,m_5} \frac{1}{[p^2 - m_4^2]^{\nu_4} [q^2 - m_5^2]^{\nu_5} [(k-p)^2 - M_1^2]^{\nu_1} [(k-q)^2 - m_2^2]^{\nu_2}}; \quad (12)$$

(ii)  $\gamma = \{1235\} \Rightarrow$

$$\int \int d^n p d^n q \frac{1}{[p^2 - m_4^2]^{\nu_4}} \times \mathcal{T}_{k,p,m_2,m_5} \frac{1}{[(p-q)^2 - M_3^2]^{\nu_3} [q^2 - m_5^2]^{\nu_5} [(k-p)^2 - M_1^2]^{\nu_1} [(k-q)^2 - m_2^2]^{\nu_2}}; \quad (13)$$

(iii)  $\gamma = \{134\} \Rightarrow$

$$\int \int d^n p d^n q \frac{1}{[(k-q)^2 - m_2^2]^{\nu_2} [q^2 - m_5^2]^{\nu_5}} \times \mathcal{T}_{k,q,m_4} \frac{1}{[(k-p)^2 - M_1^2]^{\nu_1} [(p-q)^2 - M_3^2]^{\nu_3} [p^2 - m_4^2]^{\nu_4}}; \quad (14)$$

(iv)  $\gamma = \{13\} \Rightarrow$

$$\int \int d^n p d^n q \frac{1}{[p^2 - m_4^2]^{\nu_4} [(k-q)^2 - m_2^2]^{\nu_2} [q^2 - m_5^2]^{\nu_5}} \times \mathcal{T}_{k,p,q} \frac{1}{[(k-p)^2 - M_1^2]^{\nu_1} [(p-q)^2 - M_3^2]^{\nu_3}}. \quad (15)$$

*Case 2.* Four subgraphs contribute:

(i)  $\gamma = \Gamma \Rightarrow$

$$\int \int d^n p d^n q \frac{1}{[(p-q)^2 - M_3^2]^{\nu_3}} \times \mathcal{T}_{k,m_1,m_2,m_4,m_5} \frac{1}{[p^2 - m_4^2]^{\nu_4} [q^2 - m_5^2]^{\nu_5} [(k-p)^2 - m_1^2]^{\nu_1} [(k-q)^2 - m_2^2]^{\nu_2}}; \quad (16)$$

(ii)  $\gamma = \{134\} \Rightarrow$

$$\int \int d^n p d^n q \frac{1}{[(k-q)^2 - m_2^2]^{\nu_2} [q^2 - m_5^2]^{\nu_5}} \times \mathcal{T}_{k,q,m_1,m_4} \frac{1}{[(k-p)^2 - m_1^2]^{\nu_1} [(p-q)^2 - M_3^2]^{\nu_3} [p^2 - m_4^2]^{\nu_4}}; \quad (17)$$

(iii)  $\gamma = \{235\}$  contribution can be obtained from the previous one by the permutation  $1 \leftrightarrow 2, 4 \leftrightarrow 5$ ;

(iv)  $\gamma = \{3\} \Rightarrow$

$$\int \int d^n p d^n q \frac{1}{[(k-p)^2 - m_1^2]^{\nu_1} [(k-q)^2 - m_2^2]^{\nu_2} [p^2 - m_4^2]^{\nu_4} [q^2 - m_5^2]^{\nu_5}} \mathcal{T}_{p,q} \frac{1}{[(p-q)^2 - M_3^2]^{\nu_3}}. \quad (18)$$

All the formulae (6)–(18), along with eq. (5), can be used for any values of the space-time dimension  $n$  and the powers of propagators  $\nu_i$ . Note that contributions from different subgraphs possess ultraviolet and/or infrared poles in  $\varepsilon$ . Cancellation of these poles in the sum (when a finite integral is considered) turns out to be a very strong check of the calculational procedure<sup>8</sup>.

Studying eqs. (6)–(18) shows that, after partial fractioning is performed wherever necessary, the following types of contributions can occur in the expressions for the coefficients of the small-threshold expansion (in situations with two-particle small thresholds):

- (a) two-loop vacuum diagrams with two (or one) large-mass lines and one (or two) massless lines;
- (b) products of a one-loop massive diagram (with small masses and external momentum  $k$ ) and a one-loop massive tadpole;
- (c) products of two one-loop massive diagrams with external momentum  $k$ .

The contributions of type (a) are discussed in detail in Appendix A of [17]. The one-loop two-point functions involving small masses of internal lines (occurring in the contributions of types (b) and (c)) should be calculated exactly, and they are “responsible” for describing the two-particle threshold irregularities. Some technical issues related to the calculation of these contributions are collected in Appendix A.

### 3. Analytic results

To get analytic results for the terms of the expansion describing the small-threshold behaviour, we have used the REDUCE system for analytical calculations [22]. The constructed algorithm works for any (integer) powers of the propagators and can be applied for both convergent and divergent diagrams.

As an example, in this section we consider the diagram in Fig. 1a with unit powers of propagators, the so-called “master” diagram. The corresponding scalar integral is involved in many interesting physical applications. If all five  $\nu_i$  are equal to one the corresponding diagram is finite<sup>9</sup> as  $n \rightarrow 4$ , and we shall calculate the corresponding results in four dimensions. The algorithm makes it also possible to consider higher terms of the expansion in  $\varepsilon = \frac{1}{2}(4 - n)$ , or even results for arbitrary  $n$ .

Let us write the “master” integral as<sup>10</sup>

$$J(m_1, m_2, m_3, m_4, m_5; k) \equiv J(1, 1, 1, 1, 1; m_1, m_2, m_3, m_4, m_5; k) = -\pi^4 \sum_{j=0}^{\infty} S_j, \quad (19)$$

where  $S_j$  are the terms corresponding to our expansion. For a given  $j$ , the term  $S_j$  is a sum of all contributions of the order  $(k^2)^{j_0} \prod (m_i^2)^{j_i}$  (where the product is taken over

<sup>8</sup>See also a discussion in ref. [17], p. 541.

<sup>9</sup>To be precise, there is a very special case when this “master” integral becomes infrared-divergent: when there is a two-particle *zero*-threshold and, in addition,  $k^2 = 0$ . The exact result for this case can be easily obtained by a direct calculation.

<sup>10</sup>In eq. (19),  $m_i$  may correspond to either small or large masses. In the expressions for the  $S_j$  presented below, we distinguish the small masses and the large masses by denoting them as  $m_j$  and  $M_j$ , respectively (as in section 2).



all lines with small masses) with  $j_0 + \sum j_i = j$ . Since  $k^2 J$  is dimensionless, so should be  $k^2 S_j$ . Therefore, the  $S_j$  should involve the corresponding powers of the large masses in the denominators. The numerators of  $S_j$  may also contain polynomials in the large masses, logarithms of the ratios of the masses and some functions corresponding to the contributions (a), (b), (c) described at the end of section 2. Note that (in the case when there are no zero thresholds) we should not get the terms like  $\ln(-k^2)$  producing a cut in the  $k^2$  plane starting from the origin.

The contributions of type (a) are expressed through the following dimensionless function of two variables:

$$\mathcal{H}(M_1^2, M_2^2) = 2\text{Li}_2\left(1 - \frac{M_1^2}{M_2^2}\right) + \frac{1}{2} \ln^2\left(\frac{M_1^2}{M_2^2}\right). \quad (20)$$

It is antisymmetric in its arguments and therefore vanishes when  $M_1^2 = M_2^2$ . This function is connected with the finite part of the two-loop vacuum integral with two general masses and one zero mass (for details of this definition, see Appendix A of [16]). In the contributions considered in this paper the  $\mathcal{H}$  function depends on the large masses only.

The contributions of types (b) and (c) are expressed through the function  $\tau(m_1, m_2; k^2)$  which is related to the finite part of the one-loop self-energy with general masses (see eq. (52)). It is well-known that the  $\tau$  function can be expressed in terms of logarithms and square roots (see e.g. in [23]). By using the notation

$$\Delta(m_1^2, m_2^2, k^2) \equiv 4m_1^2 m_2^2 - (k^2 - m_1^2 - m_2^2)^2 \quad (21)$$

for the “triangle” Källen function<sup>11</sup>, the result for the  $\tau$  function can be presented as

$$\tau(m_1, m_2; k^2) = \frac{1}{2k^2} \left\{ \sqrt{-\Delta} \ln \frac{k^2 - m_1^2 - m_2^2 - \sqrt{-\Delta}}{k^2 - m_1^2 - m_2^2 + \sqrt{-\Delta}} + (m_1^2 - m_2^2) \ln \frac{m_2^2}{m_1^2} + i\pi \sqrt{-\Delta} \theta(k^2 - (m_1 + m_2)^2) \right\}, \quad (22)$$

where  $\Delta \equiv \Delta(m_1^2, m_2^2, k^2)$ . In our contributions, the  $\tau$  function depends on the small masses only. In fact, it contains the main information about the small-threshold behaviour at two-particle thresholds. The  $\theta$  term in the braces yields an imaginary part in the region beyond the physical threshold (i.e. for  $k^2 > (m_1 + m_2)^2$ ). This is exactly the point where the cut in the complex  $k^2$  plane starts. Further properties of one-loop self-energy diagrams and the  $\tau$  function are discussed in Appendix A.

Now, let us present explicit results for some lowest terms of the expansion (19).

*Case 1.* For the case of general masses, the  $S_0$  contribution to eq. (19) is given by

$$S_0 = \frac{1}{4(M_1^2 - M_3^2)(M_4^2 - M_3^2)(M_1^2 - M_4^2)} \times \left\{ -4 \left( \tau(m_2, m_5; k^2) + \frac{1}{2} \ln \frac{M_1^2 M_4^2}{m_2^2 m_5^2} + 2 \right) \left[ M_1^2 (M_4^2 - M_3^2) \ln \frac{M_1^2}{M_3^2} - M_4^2 (M_1^2 - M_3^2) \ln \frac{M_4^2}{M_3^2} \right] \right.$$

<sup>11</sup>The function (21) is totally symmetric with respect to all its arguments. It vanishes at the threshold,  $k^2 = (m_1 + m_2)^2$ , and at the pseudo-threshold,  $k^2 = (m_1 - m_2)^2$ .

$$\begin{aligned}
& +(M_1^2 - M_3^2)(M_4^2 - M_3^2) \ln \frac{M_1^2}{M_4^2} \left( \ln \frac{M_1^2}{M_3^2} + \ln \frac{M_4^2}{M_3^2} \right) - 2M_3^2(M_1^2 - M_4^2) \ln \frac{M_1^2}{M_3^2} \ln \frac{M_4^2}{M_3^2} \\
& + 2(M_1^2 + M_3^2)(M_4^2 - M_3^2) \mathcal{H}(M_1^2, M_3^2) - 2(M_4^2 + M_3^2)(M_1^2 - M_3^2) \mathcal{H}(M_4^2, M_3^2) \Big\} , \quad (23)
\end{aligned}$$

We note that this result is very similar to the corresponding contribution for the zero-threshold case,  $C_0$  (see eq. (21) of ref. [17]). Indeed, in the massless limit we get (cf. eq. (59))

$$\tau(m_1, m_2; k^2) \Big|_{m_1, m_2 \rightarrow 0} = \frac{1}{2} \left( \ln \left( -\frac{m_1^2}{k^2} \right) + \ln \left( -\frac{m_2^2}{k^2} \right) \right). \quad (24)$$

Therefore, the massless limit of the brackets in (23) containing the  $\tau$  function yields the same combination involving  $\ln(-k^2)$  as in eq. (21) of [17], while the remaining terms coincide. The next term,  $S_1$ , involves the  $k^2$ ,  $m_2^2$  and  $m_5^2$  contributions (we present the corresponding result in Appendix B). Note that in this case the coefficient of  $k^2$  gives (in the massless limit, i.e. using eq. (24)) the corresponding zero-threshold coefficient  $C_1$  (see eq. (22) of [17]), while the  $m_2^2$  and  $m_5^2$  contributions have no analogy with the zero-threshold case.

We have also obtained higher terms of the expansion (19) with five different masses ( $S_2$  and  $S_3$ ), but they are more cumbersome, and we do not present them here.

When some large masses are equal, one can use this fact from the very beginning, when calculating contributions of the corresponding subgraphs. Another possibility is to consider, with due care (since both denominators and numerators vanish in this limit), the corresponding limit of the expressions with different masses. On one hand, the second option is a good way to check the consistency of the obtained results. On the other hand, the first way (i.e. having fewer different masses from the very beginning) simplifies the calculation and therefore makes it possible to obtain expressions for higher terms.

For example, we get the following expressions for the first two coefficients corresponding to case 1 with  $M_1 = M_4 \equiv M$ :

$$\begin{aligned}
S_0 = -\frac{1}{(M^2 - M_3^2)^2} \Big\{ & \left( \tau(m_2, m_5; k^2) + \frac{1}{2} \ln \frac{M^4}{m_2^2 m_5^2} \right) \left( M_3^2 \ln \frac{M_3^2}{M^2} + M^2 - M_3^2 \right) \\
& + M_3^2 \left( \mathcal{H}(M^2, M_3^2) + \frac{1}{2} \ln^2 \frac{M_3^2}{M^2} \right) + 2(M^2 - M_3^2) \Big\}, \quad (25)
\end{aligned}$$

$$\begin{aligned}
S_1 = & -\frac{1}{12M^2(M^2 - M_3^2)^4} \\
& \times \Big\{ \left( \tau(m_2, m_5; k^2) + \frac{1}{2} \ln \frac{M^4}{m_2^2 m_5^2} \right) \left( 6M^2 M_3^2 \ln \frac{M^2}{M_3^2} \left( k^2 M_3^2 - (m_2^2 + m_5^2)(2M^2 + M_3^2) \right) \right. \right. \\
& \quad \left. \left. + (M^2 - M_3^2) \left( k^2(M^4 - 5M^2 M_3^2 - 2M_3^4) + 3(m_2^2 + m_5^2)M^2(M^2 + 5M_3^2) \right) \right) \right. \\
& \quad \left. - 6M^2 M_3^2 \left( \mathcal{H}(M^2, M_3^2) + \frac{1}{2} \ln^2 \frac{M_3^2}{M^2} \right) \left( k^2 M_3^2 - 2(m_2^2 + m_5^2)(2M^2 + M_3^2) \right) \right. \\
& \quad \left. + 3M^2 \left( m_2^2 \ln \frac{m_2^2}{M^2} + m_5^2 \ln \frac{m_5^2}{M^2} \right) \left( 2M_3^2(2M^2 + M_3^2) \ln \frac{M^2}{M_3^2} - (M^2 - M_3^2)(M^2 + 5M_3^2) \right) \right\}
\end{aligned}$$

$$\begin{aligned}
& +2M_3^2 \ln \frac{M^2}{M_3^2} \left( k^2(5M^4 - M_3^4) + 12(m_2^2 + m_5^2)M^2 M_3^2 \right) \\
& + (M^2 - M_3^2) \left( 6(m_2^2 + m_5^2)M^2(M^2 + 7M_3^2) - k^2(M^4 + 17M^2 M_3^2 + 2M_3^4) \right) \Big\}. \quad (26)
\end{aligned}$$

For this case, we have also obtained the  $S_2$ ,  $S_3$  and  $S_4$  terms.

If all large masses are equal (i.e.  $M_1 = M_3 = M_4 \equiv M$ ), the expressions become very simple:

$$S_0 = -\frac{1}{2M^2} \left( \tau(m_2, m_5; k^2) + \frac{1}{2} \ln \frac{M^4}{m_2^2 m_5^2} + 3 \right), \quad (27)$$

$$\begin{aligned}
S_1 = -\frac{1}{144M^4} \Big\{ & 6(k^2 + m_2^2 + m_5^2) \left( \tau(m_2, m_5; k^2) + \frac{1}{2} \ln \frac{M^4}{m_2^2 m_5^2} \right) \\
& + 6 \left( m_2^2 \ln \frac{M^2}{m_2^2} + m_5^2 \ln \frac{M^2}{m_5^2} \right) + 9k^2 + 26(m_2^2 + m_5^2) \Big\}. \quad (28)
\end{aligned}$$

For this case, we have obtained the terms up to (and including)  $S_6$ .

*Case 1a.* For general masses, the lowest case 1a contributions to (19) are

$$S_0 = -\frac{1}{M_1^2 - M_4^2} \ln \frac{M_1^2}{M_4^2} \left( \tau(m_2, m_5; k^2) + \frac{1}{2} \ln \frac{M_1^2 M_4^2}{m_2^2 m_5^2} + 2 \right), \quad (29)$$

$$\begin{aligned}
S_1 = & -\frac{1}{4M_1^2 M_4^2 (M_1^2 - M_4^2)^3} \Big\{ 2 \left( \tau(m_2, m_5; k^2) + \frac{1}{2} \ln \frac{M_1^2 M_4^2}{m_2^2 m_5^2} + 2 \right) \\
& \times \left[ M_1^2 M_4^2 \left( ((M_1^2 + M_4^2)k^2 + (M_1^2 - M_4^2)(m_2^2 - m_5^2)) \ln \frac{M_1^2}{M_4^2} - 2(M_1^2 - M_4^2)k^2 \right) \right. \\
& \quad \left. + (M_1^2 - M_4^2)^2 \left( 2m_3^2 \left( M_1^2 \ln \frac{m_3^2}{M_4^2} - M_4^2 \ln \frac{m_3^2}{M_1^2} \right) - M_4^2 m_2^2 + M_1^2 m_5^2 \right) \right] \\
& + M_1^2 M_4^2 (M_1^2 - M_4^2) \ln \frac{M_1^2}{M_4^2} \left( m_2^2 \ln \frac{m_2^4}{M_1^2 M_4^2} - m_5^2 \ln \frac{m_5^4}{M_1^2 M_4^2} - 2m_2^2 + 2m_5^2 \right) \\
& - (M_1^2 - M_4^2)^2 \left[ M_1^2 m_2^2 \ln \frac{m_2^4}{M_1^2 M_4^2} - M_4^2 m_5^2 \ln \frac{m_5^4}{M_1^2 M_4^2} + (m_2^2 + m_5^2)(M_1^2 + M_4^2) \ln \frac{M_1^2}{M_4^2} \right. \\
& \quad \left. + 2m_3^2 \ln \frac{M_1^2}{M_4^2} \left( M_1^2 \ln \frac{m_3^2}{M_4^2} + M_4^2 \ln \frac{m_3^2}{M_1^2} \right) + 4m_3^2 \left( M_1^2 \ln \frac{m_3^2}{M_4^2} - M_4^2 \ln \frac{m_3^2}{M_1^2} \right) \right. \\
& \quad \left. + (M_1^2 - M_4^2) \left( k^2 - m_2^2 - m_5^2 + 4m_3^2 \left( \frac{1}{3}\pi^2 - 1 \right) \right) - 2M_1^2 m_2^2 + 2M_4^2 m_5^2 \right] \Big\}. \quad (30)
\end{aligned}$$

For the general case 1a, we have also obtained the terms up to (and including)  $S_5$ .

If  $M_1 = M_4 \equiv M$ , the corresponding results are

$$S_0 = -\frac{1}{M^2} \left( \tau(m_2, m_5; k^2) + \frac{1}{2} \ln \frac{M^4}{m_2^2 m_5^2} + 2 \right), \quad (31)$$

$$S_1 = -\frac{1}{12M^4} \left\{ \left( \tau(m_2, m_5; k^2) + \frac{1}{2} \ln \frac{M^4}{m_2^2 m_5^2} \right) \left( k^2 + 3(m_2^2 + m_5^2 + 4m_3^2) + 12m_3^2 \ln \frac{m_3^2}{M^2} \right) \right. \\ \left. - 3 \left( m_2^2 \ln \frac{m_2^2}{M^2} + m_5^2 \ln \frac{m_5^2}{M^2} \right) - k^2 + 6(m_2^2 + m_5^2 + 4m_3^2) - 4\pi^2 m_3^2 \right\}. \quad (32)$$

Case 1b. The results for the lowest terms,  $S_0$  and  $S_1$ , are given by

$$S_0 = -\frac{1}{2M_1^2(M_1^2 - M_3^2)} \left\{ 2M_1^2 \ln \frac{M_1^2}{M_3^2} \left( \tau(m_2, m_5; k^2) + \frac{1}{2} \ln \frac{M_3^4}{m_2^2 m_5^2} + 2 \right) \right. \\ \left. - (M_1^2 + M_3^2) \left( \mathcal{H}(M_1^2, M_3^2) - \frac{1}{2} \ln^2 \frac{M_1^2}{M_3^2} \right) - \frac{1}{3} \pi^2 (M_1^2 - M_3^2) \right\}, \quad (33)$$

$$S_1 = \frac{1}{4M_1^4 M_3^2 (M_1^2 - M_3^2)^3} \left\{ -2M_1^2 \left( \tau(m_2, m_5; k^2) + \frac{1}{2} \ln \frac{M_3^4}{m_2^2 m_5^2} + 2 \right) \right. \\ \times \left[ M_1^2 M_3^2 \left( ((M_1^2 - M_3^2)(k^2 - m_5^2) + (M_1^2 + M_3^2)m_2^2) \ln \frac{M_1^2}{M_3^2} - 2(M_1^2 - M_3^2)m_2^2 \right) \right. \\ \left. + (M_1^2 - M_3^2)^2 \left( 2m_4^2 \left( M_1^2 \ln \frac{m_4^2}{M_3^2} - M_3^2 \ln \frac{m_4^2}{M_1^2} \right) - M_3^2 k^2 + M_1^2 m_5^2 \right) \right] \\ + M_3^2 \left( \mathcal{H}(M_1^2, M_3^2) - \frac{1}{2} \ln^2 \frac{M_1^2}{M_3^2} \right) \left[ (M_1^2 - M_3^2)^2 (M_1^2 + M_3^2) (k^2 + 2m_4^2) \right. \\ \left. + 2M_1^2 M_3^2 (M_1^2 - M_3^2) k^2 + 4M_1^4 M_3^2 (m_2^2 + m_5^2) \right] \\ + (M_1^2 - M_3^2) k^2 \left( 2M_1^2 M_3^2 (M_1^2 + M_3^2) \ln \frac{M_1^2}{M_3^2} + M_1^4 (M_1^2 - 3M_3^2) + \frac{1}{3} \pi^2 M_3^2 (M_1^2 - M_3^2)^2 \right) \\ + 4(M_1^2 - M_3^2)^3 m_4^2 \left( M_1^2 \ln \frac{m_4^2}{M_3^2} - M_1^2 + \frac{1}{6} \pi^2 (2M_1^2 + M_3^2) \right) \\ + 2M_1^4 \left( -(M_1^2 - M_3^2) m_2^2 \ln \frac{m_2^2}{M_3^2} \left( M_3^2 \ln \frac{M_1^2}{M_3^2} - M_1^2 + M_3^2 \right) \right. \\ \left. + M_3^2 m_5^2 \ln \frac{m_5^2}{M_3^2} \left( (M_1^2 + M_3^2) \ln \frac{M_1^2}{M_3^2} - 2M_1^2 + 2M_3^2 \right) \right) \\ + 2M_1^4 M_3^2 \ln \frac{M_1^2}{M_3^2} \left( (2M_1^2 + M_3^2) m_2^2 + M_3^2 (k^2 + m_5^2) \right) \\ \left. + M_1^4 (M_1^2 - M_3^2) \left( (5M_3^2 - 3M_1^2) m_2^2 + (7M_3^2 - M_1^2) m_5^2 \right) \right\}. \quad (34)$$

For the general case 1b, we have also obtained the  $S_2$ ,  $S_3$  and  $S_4$  terms.

If  $M_1 = M_3 \equiv M$ , the results for  $S_0$  and  $S_1$  are

$$S_0 = -\frac{1}{M^2} \left( \tau(m_2, m_5; k^2) + \frac{1}{2} \ln \frac{M^4}{m_2^2 m_5^2} + \frac{1}{6} \pi^2 + 4 \right), \quad (35)$$

$$S_1 = -\frac{1}{12M^4} \left\{ \left( \tau(m_2, m_5; k^2) + \frac{1}{2} \ln \frac{M^4}{m_2^2 m_5^2} + 1 \right) \left[ 3k^2 + m_2^2 + 3m_5^2 + 12m_4^2 \left( \ln \frac{m_4^2}{M^2} + 1 \right) \right] \right. \\ \left. - 3m_2^2 \ln \frac{m_2^2}{M^2} - m_5^2 \ln \frac{m_5^2}{M^2} - k^2 (\pi^2 - 10) - 6m_4^2 (\pi^2 - 8) + \frac{11}{3} (m_2^2 + m_5^2) \right\}. \quad (36)$$

Before proceeding to the case 2, let us discuss a connection between the results for the cases 1 and 1a,b. All of them correspond to the same small-threshold configuration (one small two-particle threshold), and one could expect that there should be a transition between explicit expressions for the terms of the expansion (19) for the case 1 and those for the cases 1a and 1b. In the zero-threshold case [17], what we needed was just to put one more mass (in the coefficients  $C_j$  for the case 1) to be zero, and we arrived at the results for the corresponding case 1a and case 1b coefficients. When we deal with small (but non-zero) thresholds, the situation is more tricky. We need to consider one more mass to be small, but now the contributions proportional to this small mass will go to the higher  $S_j$ . In addition, we need to expand the denominators involving this small mass as well as the corresponding  $\mathcal{H}$  function(s) involving a small and a large mass. Expansion of the denominators is trivial, while the  $\mathcal{H}$  function can be expanded using the following formula:

$$\mathcal{H}(M^2, m^2) = 2 \sum_{l=1}^{\infty} \frac{1}{l^2} \left( \frac{m^2}{M^2} \right)^l - 2 \ln \frac{m^2}{M^2} \sum_{l=1}^{\infty} \frac{1}{l} \left( \frac{m^2}{M^2} \right)^l - \frac{1}{2} \ln^2 \frac{m^2}{M^2} - \frac{1}{3} \pi^2. \quad (37)$$

So, if one wants to show the correspondence between the case 1 and the cases 1a,b, one should not consider the  $S_j$  themselves but to collect the terms of required order in the whole sum (19) instead. In such a way, we have checked that all available terms for the case 1 produce the corresponding terms for the cases 1a and 1b. On one hand, the described additional “subexpansion” in a small mass corresponds, in some sense, to additional subgraphs appearing in the cases 1a and 1b as compared with the case 1, see Fig. 2. On the other hand, if one is interested in cases 1a and 1b, it is better to consider these cases from the very beginning rather than get results from the case 1 (since more terms are available for the cases 1a and 1b).

*Case 2.* The results for the  $S_0$  and  $S_1$  terms of the expansion (19) are

$$S_0 = -\frac{1}{M_3^2} \left\{ \left( \tau(m_1, m_4; k^2) + \frac{1}{2} \ln \frac{M_3^4}{m_1^2 m_4^2} + 1 \right) \left( \tau(m_2, m_5; k^2) + \frac{1}{2} \ln \frac{M_3^4}{m_2^2 m_5^2} + 1 \right) + \frac{1}{3} \pi^2 - 1 \right\}, \quad (38)$$

$$\begin{aligned} S_1 = \frac{1}{2M_3^4} & \left\{ (k^2 - m_1^2 - m_2^2 - m_4^2 - m_5^2) \right. \\ & \times \left( \tau(m_1, m_4; k^2) + \frac{1}{2} \ln \frac{M_3^4}{m_1^2 m_4^2} - \frac{1}{2} \right) \left( \tau(m_2, m_5; k^2) + \frac{1}{2} \ln \frac{M_3^4}{m_2^2 m_5^2} - \frac{1}{2} \right) \\ & + \left( \tau(m_1, m_4; k^2) + \frac{1}{2} \ln \frac{M_3^4}{m_1^2 m_4^2} - \frac{1}{2} \right) \left( m_2^2 \ln \frac{m_2^2}{M_3^2} + m_5^2 \ln \frac{m_5^2}{M_3^2} - \frac{3}{2} (m_2^2 + m_5^2) + k^2 \right) \\ & + \left( \tau(m_2, m_5; k^2) + \frac{1}{2} \ln \frac{M_3^4}{m_2^2 m_5^2} - \frac{1}{2} \right) \left( m_1^2 \ln \frac{m_1^2}{M_3^2} + m_4^2 \ln \frac{m_4^2}{M_3^2} - \frac{3}{2} (m_1^2 + m_4^2) + k^2 \right) \\ & - \left( \frac{2}{3} \pi^2 - \frac{9}{2} \right) (m_1^2 + m_2^2 + m_4^2 + m_5^2 - \frac{1}{2} k^2) \\ & \left. + \frac{(m_1^2 - m_4^2)(m_2^2 - m_5^2)}{k^2} (\tau(m_1, m_4; k^2) - \tau(m_1, m_4; 0)) (\tau(m_2, m_5; k^2) - \tau(m_2, m_5; 0)) \right\}, \quad (39) \end{aligned}$$

where the value of the  $\tau$  function at  $k^2 = 0$  is

$$\tau(m_1, m_2; 0) = -1 - \frac{1}{2} \frac{m_1^2 + m_2^2}{m_1^2 - m_2^2} \ln \frac{m_1^2}{m_2^2}. \quad (40)$$

For the general case 2, we have obtained the terms of the expansion (19) up to (and including)  $S_6$ .

An interesting feature of eq. (39) is the appearance of  $k^2$  in the denominator. For higher terms, higher powers of  $k^2$  in the denominator occur<sup>12</sup>, e.g.  $(k^2)^2$  for  $S_2$ , etc. For general masses, the powers of  $k^2$  in the denominator can be cancelled by considering the Taylor expansion of the  $\tau$  function in  $k^2$ . The simplest example can be seen in eq. (39).

To conclude this section, we would like to note that we have successfully compared the massless limit of the small-threshold expansion with *all* coefficients from the four first columns (corresponding to the cases 1, 1a, 1b and 2, respectively) of Table 1 presented on p. 545 of ref. [17].

#### 4. Numerical results

Let us show how the small-threshold expansion can be applied to obtain approximate numerical results for self-energy diagrams. We shall use as examples a number of mass configurations corresponding to diagrams occurring in the Standard Model. We take the masses of the  $W$  boson, and of the charmed, bottom and top quarks to be<sup>13</sup>:

$$M_W = 80 \text{ GeV}, \quad m_c = 1.5 \text{ GeV}, \quad m_b = 5 \text{ GeV}, \quad m_t = 174 \text{ GeV}. \quad (41)$$

In this section, we shall consider the following approximations to the “master” integral (19) (corresponding to Fig. 1a):

$$J^{(N)} = -\pi^4 \sum_{j=0}^N S_j. \quad (42)$$

As in ref. [17], the first example corresponds to a diagram containing a top-bottom loop to which two  $W$  bosons are attached. This diagram contributes to the self-energy of the photon and the  $Z$  boson. The corresponding scalar integral is

$$J(M_W, m_b, m_t, M_W, m_b; k). \quad (43)$$

If we consider  $m_b$  as a small mass, and  $m_t$  and  $M_W$  as large masses<sup>14</sup>, this diagram has one small two-particle threshold at  $k^2 = 4m_b^2$  and therefore it belongs to case 1. In Fig. 3 the approximations  $J^{(N)}$  defined by eq. (42) are shown as curves, and, for comparison, values of  $J$  obtained by numerical integration [7] are shown as crosses. The position of the lowest “large” threshold, at  $k^2 = 4M_W^2$  in this example, is indicated by a vertical line.

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<sup>12</sup>The corresponding contributions vanish when either  $m_1^2 = m_4^2$  or  $m_2^2 = m_3^2$ .

<sup>13</sup>In this section, we adopt “standard” notation for the masses of physical particles: the quark masses are denoted with small  $m$  whereas the capital  $M$  is used for the vector boson masses. To avoid any confusion with the notation of sections 2 and 3, we shall explicitly state, for each concrete example, which masses are considered to be small and large.

<sup>14</sup>Note that in ref. [17] we have considered the situation when  $m_b = 0$ . The situation for non-zero  $m_b$  was discussed as a subject to be investigated in the future. This paper solves the problem posed in [17].

One can see that the expansion converges all the way up to the first large threshold at  $k^2 = 25600 \text{ GeV}^2$ .

In fact, the behaviour for  $4m_b^2 \ll k^2 < 4M_W^2$  is described as well as in [17]. The main difference is in the region of small  $k^2$  (in particular, around the  $4m_b^2$  threshold) which could not be described by the zero-threshold expansion [17]. In the scale of Fig. 3, the details of the small-threshold behaviour cannot be seen so that we use a different pair of plots in Fig. 4 to illustrate the behaviour in the region of small momenta, including the threshold at  $k^2 = 4m_b^2 = 100 \text{ GeV}^2$ . One can see that even the zero order approximation  $J^{(0)}$  is very good and that it is sufficient to take  $J^{(1)}$  to get result with a high precision. We do not present other approximations (up to  $N = 4$ ) because they give a precision better than the numerical program does (and one cannot distinguish them in the plot). In our approximations, the irregularities at the small threshold are reproduced by the  $\tau$  function (22). In particular, the imaginary part is zero below the threshold and behaves like a square root in the region above the threshold.

In fact, one can get a better description of the behaviour of the same diagram (43) in the region around the  $k^2 = 4M_W^2$  threshold by considering  $M_W$  as a small mass. In this case, the only remaining large mass is  $m_t$ . Thus, it belongs to case 2, in our terminology. According to our results, the situation in the region of small momenta is more or less the same: the first two approximations happen to be good enough (but not so good as when the diagram is considered as case 1). Our approximations in the region up to the large three-particle threshold (at  $k^2 = (m_t + M_W + m_b)^2$ ) are shown in Fig. 5. One can see that our approximations describe the behaviour around the  $4M_W^2$  threshold and, moreover, they work even beyond this second threshold<sup>15</sup>. Thus, just by treating in another way the available massive parameters it is possible to characterize analytically the first diagram in a large region of momenta.

As second example we consider the integral

$$J(m_b, m_c, M_W, m_b, m_c; k) \quad (44)$$

which has two small two-particle thresholds, at  $k^2 = 4m_c^2$  and  $k^2 = 4m_b^2$ , and therefore belongs to case 2.

The behaviour up to the first large threshold is rather similar to the previous examples so we do not present the corresponding plot. The behaviour at small momenta including both small thresholds can be seen in Fig. 6. A rather tricky behaviour in the region around these two small thresholds is very well reproduced by taking the lowest analytical approximations,  $J^{(0)}$  (for the imaginary part) or  $J^{(1)}$  (for the real part).

All the examples illustrate that the small-threshold expansion provides approximations which perfectly describe the small-threshold behaviour. Moreover, they are at least as accurate as numerical integration in a large part of the region of convergence and can be evaluated much faster.

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<sup>15</sup>The convergence between the  $4M_W^2$  and the  $(m_t + M_W + m_b)^2$  thresholds is not so good since the  $M_W/m_t$  ratio is not very small.

## 5. Conclusions

In this paper we have studied the behaviour of two-loop self-energy diagrams in the situation when the external momentum and some of the masses are small with respect to the large masses. All configurations with small two-particle thresholds have been considered. By use of explicit formulae for the terms of asymptotic expansions in the large mass limit, we presented an analytic approach to calculating these diagrams by keeping the first few terms of the expansion.

By taking some complicated cases (corresponding to diagrams with different masses occurring in the Standard Model) as examples, we compared our results with those of a numerical integration program based on the algorithm of ref. [7] (see also in [9]). We have shown that our analytical approximations work very well in the region of small momenta that includes all the small thresholds. Moreover, the small-threshold expansion converges up to a region close to the first large threshold. Unless  $k^2$  is close to the large threshold, only a few terms are needed to obtain accurate results. This comparison can also be considered as a check of the numerical program.

Thus we have solved the problem of the threshold behaviour of two-loop two-point functions for the small two-particle thresholds. It is interesting that it was possible to do this just by using the expansion in the large masses. The main idea was to avoid putting any conditions on relative values of the external momentum squared and small masses. To our knowledge, no efficient algorithms for describing the non-zero threshold behaviour of two-loop diagrams with arbitrary masses were available so far<sup>16</sup>. A nice feature of the presented approach is that all two-particle threshold irregularities of two-loop diagrams are incorporated in the function corresponding to one-loop diagrams, the  $\tau$  function.

A natural extension of this work could be connected with cases 3 and 4 (according to the classification given in Section 2), when three-particle small thresholds arise. To carry out a similar program for those cases we can apply the same technique of asymptotic expansion in large mass(es). In particular, in eq. (5) one should sum over the subgraphs obeying the same conditions as in the cases considered. For example, for case 3 (with small masses  $m_2, m_3$  and  $m_4$ ) the following five subgraphs  $\gamma$  contribute to the sum (5):  $\Gamma \equiv \{12345\}, \{1235\}, \{1345\}, \{1245\}$  and  $\{15\}$ . For case 4 (when all the masses except  $M_1$  are small), four subgraphs contribute:  $\Gamma, \{1245\}, \{134\}$  and  $\{1\}$ . However, in these cases the small-threshold behaviour at three-particle threshold is defined by the functions corresponding to the sunset diagram (with three propagators) and the diagram with four propagators (e.g. the integral (1) with  $\nu_1 = 0$ ). Unfortunately, sufficient *analytic* information about threshold behaviour of these diagrams is at the moment not available.

The algorithm presented in this paper can be also extended to the three-point two-loop diagrams with different masses.

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<sup>16</sup>A possible exception may be related to an interesting approach considered in ref. [20] where a two-loop diagram with one non-zero mass parameter was studied as an example. It is not clear, however, how efficiently the method of [20] will work for two-loop diagrams with different non-zero masses, since the coefficients of the expansion may require results for the threshold values of two-loop integrals (which, for the general case, are not known analytically).



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## Appendix A. One-loop two-point integrals with masses

In this appendix, we collect some relevant results for the integrals

$$J(\nu_1, \nu_2; m_1, m_2) = \int \frac{d^n q}{[q^2 - m_1^2]^{\nu_1} [(k - q)^2 - m_2^2]^{\nu_2}}. \quad (45)$$

Let us introduce dimensionless quantities

$$x \equiv \frac{m_1^2}{k^2}, \quad y \equiv \frac{m_2^2}{k^2}, \quad (46)$$

$$\lambda(x, y) = \sqrt{(1 - x - y)^2 - 4xy}. \quad (47)$$

It is easy to see that the Källen function  $\Delta$ , eq. (21), is related to  $\lambda$  via

$$\Delta(m_1^2, m_2^2, k^2) = 4m_1^2 m_2^2 - (k^2 - m_1^2 - m_2^2)^2 = -(k^2)^2 \lambda^2(x, y). \quad (48)$$

Therefore,  $\lambda$  also vanishes at the threshold,  $k^2 = (m_1 + m_2)^2$ , and at the pseudo-threshold,  $k^2 = (m_1 - m_2)^2$ .

For arbitrary  $\nu_1, \nu_2$  and the space-time dimension  $n$ , the result for the integral (45) can be written as (see in [24])

$$\begin{aligned} J(\nu_1, \nu_2; m_1, m_2) &= \pi^{n/2} i^{1-n} (k^2)^{n/2 - \nu_1 - \nu_2} \\ &\times \left\{ \frac{\Gamma\left(\frac{n}{2} - \nu_1\right) \Gamma\left(\frac{n}{2} - \nu_2\right) \Gamma\left(\nu_1 + \nu_2 - \frac{n}{2}\right)}{\Gamma(\nu_1) \Gamma(\nu_2) \Gamma(n - \nu_1 - \nu_2)} \right. \\ &\quad \times F_4\left(\nu_1 + \nu_2 - \frac{n}{2}, \nu_1 + \nu_2 - n + 1; \nu_1 - \frac{n}{2} + 1, \nu_2 - \frac{n}{2} + 1 \middle| x, y\right) \\ &\quad + (-x)^{n/2 - \nu_1} \frac{\Gamma\left(\nu_1 - \frac{n}{2}\right)}{\Gamma(\nu_1)} F_4\left(\nu_2, \nu_2 - \frac{n}{2} + 1; \frac{n}{2} - \nu_1 + 1, \nu_2 - \frac{n}{2} + 1 \middle| x, y\right) \\ &\quad \left. + (-y)^{n/2 - \nu_2} \frac{\Gamma\left(\nu_2 - \frac{n}{2}\right)}{\Gamma(\nu_2)} F_4\left(\nu_1, \nu_1 - \frac{n}{2} + 1; \nu_1 - \frac{n}{2} + 1, \frac{n}{2} - \nu_2 + 1 \middle| x, y\right) \right\}, \quad (49) \end{aligned}$$

where  $F_4$  is the Appell hypergeometric function of two variables,

$$F_4(a, b; c, d | x, y) = \sum_{j=0}^{\infty} \sum_{l=0}^{\infty} \frac{(a)_{j+l} (b)_{j+l}}{(c)_j (d)_l} \frac{x^j y^l}{j! l!}, \quad (50)$$

and  $(a)_j \equiv \Gamma(a + j)/\Gamma(a)$  is the Pochhammer symbol.

For  $\nu_1 = \nu_2 = 1$ , but keeping the space-time dimension  $n \equiv 4 - 2\varepsilon$  arbitrary, the result simplifies and can be written in terms of the Gauss hypergeometric function  ${}_2F_1$ ,

$$J(1, 1; m_1, m_2) = i \pi^{2-\varepsilon} (-k^2)^{-\varepsilon} \left\{ \frac{\Gamma^2(1-\varepsilon)\Gamma(\varepsilon)}{\Gamma(2-2\varepsilon)} \lambda^{1-2\varepsilon} \right. \\ \left. -\Gamma(-1+\varepsilon) \frac{1}{2}(1+x-y-\lambda)(-x)^{-\varepsilon} {}_2F_1 \left( \begin{matrix} 1, \varepsilon \\ 2-\varepsilon \end{matrix} \middle| \frac{(1+x-y-\lambda)^2}{4x} \right) \right. \\ \left. -\Gamma(-1+\varepsilon) \frac{1}{2}(1-x+y-\lambda)(-y)^{-\varepsilon} {}_2F_1 \left( \begin{matrix} 1, \varepsilon \\ 2-\varepsilon \end{matrix} \middle| \frac{(1-x+y-\lambda)^2}{4y} \right) \right\}. \quad (51)$$

Expanding in  $\varepsilon = \frac{1}{2}(4-n)$  we get

$$J(1, 1; m_1, m_2) = i\pi^{2-\varepsilon} \Gamma(1+\varepsilon) \left\{ \frac{1}{\varepsilon} + 2 - \frac{1}{2}(\ln m_1^2 + \ln m_2^2) + \tau(m_1, m_2; k^2) + \mathcal{O}(\varepsilon) \right\}. \quad (52)$$

where the  $\tau$  function can be defined via an integral representation

$$\tau(m_1, m_2; k^2) \equiv \tau(x, y) = - \int_0^1 d\alpha \ln \left( \frac{\alpha m_1^2 + (1-\alpha)m_2^2 - \alpha(1-\alpha)k^2}{\alpha(1-\alpha)m_1 m_2} \right). \quad (53)$$

It is understood that  $k^2 \leftrightarrow k^2 + i0$ .

For  $k^2 < (m_1 - m_2)^2$  (including the Euclidean region,  $k^2 < 0$ ), the result for the  $\tau$  function can be written as

$$\tau(x, y) = \frac{1}{2}\lambda \ln \frac{1-x-y-\lambda}{1-x-y+\lambda} + \frac{1}{2}(x-y) \ln \frac{y}{x}. \quad (54)$$

This is also valid all the way up to the threshold,  $k^2 = (m_1 + m_2)^2$ . However, since  $\lambda(x, y)$  is imaginary for  $(m_1 - m_2)^2 < k^2 < (m_1 + m_2)^2$ ,  $\lambda^2 < 0$ , in this region one can also use another representation,

$$\tau(x, y) = -\sqrt{-\lambda^2} \arccos \left( \frac{x+y-1}{2\sqrt{xy}} \right) + \frac{1}{2}(x-y) \ln \frac{y}{x}. \quad (55)$$

Beyond the threshold, at  $k^2 > (m_1 + m_2)^2$ ,  $\lambda^2$  is again positive, but the  $\tau$  function obtains an imaginary part,

$$\tau(x, y) = i\pi\lambda + \frac{1}{2}\lambda \ln \frac{1-x-y-\lambda}{1-x-y+\lambda} + \frac{1}{2}(x-y) \ln \frac{y}{x}. \quad (56)$$

Note that at  $\lambda = 0$  the expression for the  $\tau$  function is

$$\tau(x, y) \Big|_{\lambda=0} = \frac{1}{2}(x-y) \ln \frac{y}{x}, \quad (57)$$

whereas at  $k^2 = 0$  we get the result (40).

When one or two masses  $m_i$  vanish, the  $\tau$  function develops a logarithmic singularity. For example, taking the limit  $m_2 \rightarrow 0$  yields the well-known result

$$\tau(m_1, m_2; k^2) \Big|_{m_2 \rightarrow 0} = \frac{1}{2} \ln \frac{m_2^2}{m_1^2} + \frac{m_1^2 - k^2}{k^2} \ln \frac{m_1^2 - k^2}{m_1^2}. \quad (58)$$

The term  $\frac{1}{2} \ln m_2^2$  cancels the corresponding term in (52). The rest is real for  $k^2 < m_1^2$ , i.e. below the threshold (in this case the threshold and the pseudo-threshold coincide). For  $k^2 > m_1^2$  (beyond the threshold), we should remember that  $k^2 \leftrightarrow k^2 + i0$  and substitute  $\ln(m_1^2 - k^2) \leftrightarrow \ln(k^2 - m_1^2) - i\pi$ .

If both masses vanish, we get

$$\tau(m_1, m_2; k^2) \Big|_{m_1, m_2 \rightarrow 0} = \frac{1}{2} \left( \ln \left( -\frac{m_1^2}{k^2} \right) + \ln \left( -\frac{m_2^2}{k^2} \right) \right). \quad (59)$$

The terms  $\frac{1}{2}(\ln m_1^2 + \ln m_2^2)$  cancel the corresponding terms in (52), while the remaining  $\ln(-k^2)$  should be understood as  $(\ln k^2 - i\pi)$  for the time-like values of the momentum. In this limit, the results of the present paper correspond to the results of zero-threshold expansion [17].

The integrals  $J(\nu_1, \nu_2; m_1, m_2)$  with higher integer values of  $\nu_1$  or  $\nu_2$  can be obtained by using the integration-by-parts technique [25]. The way is quite similar to one used in [26]. First, we get a system of two equations,

$$\begin{aligned} -2\nu_1 m_1^2 J(\nu_1 + 1, \nu_2) + \nu_2 (k^2 - m_1^2 - m_2^2) J(\nu_1, \nu_2 + 1) \\ = -(n - 2\nu_1 - \nu_2) J(\nu_1, \nu_2) + \nu_2 J(\nu_1 - 1, \nu_2 + 1), \\ \nu_1 (k^2 - m_1^2 - m_2^2) J(\nu_1 + 1, \nu_2) - 2\nu_2 m_2^2 J(\nu_1, \nu_2 + 1) \\ = -(n - \nu_1 - 2\nu_2) J(\nu_1, \nu_2) + \nu_1 J(\nu_1 + 1, \nu_2 - 1). \end{aligned} \quad (60)$$

The determinant of the matrix composed of the coefficients on the l.h.s. of this system is proportional to  $\Delta(m_1^2, m_2^2, k^2)$ , eq. (48). Solving the system, we get

$$\begin{aligned} J(\nu_1 + 1, \nu_2) = \frac{1}{\nu_1 \Delta} \left\{ \left( (k^2 - m_1^2)(n - \nu_1 - 2\nu_2) + m_2^2(n - 3\nu_1) \right) J(\nu_1, \nu_2) \right. \\ \left. - 2\nu_2 m_2^2 J(\nu_1 - 1, \nu_2 + 1) - \nu_1 (k^2 - m_1^2 - m_2^2) J(\nu_1 + 1, \nu_2 - 1) \right\}, \end{aligned} \quad (61)$$

and an analogous result for  $J(\nu_1, \nu_2 + 1)$ . By using these results, all the integrals with higher integer  $\nu$ 's can be expressed in terms of the integral  $J(1, 1; m_1, m_2)$  and the massive tadpoles.

We also need some formulae for the integrals with numerators. For the integrals (45) with one negative power of the denominator, we get

$$\begin{aligned} J(\nu, -N; m_1, m_2) &= \int \frac{d^n q}{[q^2 - m_1^2]^\nu} \left[ (k - q)^2 - m_2^2 \right]^N \\ &= \sum_{j=0}^{[N/2]} \frac{N!}{j!(N-2j)! \left(\frac{n}{2}\right)_j} (k^2)^j \int d^n q \frac{(q^2)^j (k^2 + q^2 - m_2^2)^{N-2j}}{[q^2 - m_1^2]^\nu}. \end{aligned} \quad (62)$$

Another formula we would like to present is useful for dealing with the two-loop integrals with the denominators corresponding to a product of two one-loop integrals (1), but when the numerator is a power of the scalar product of two integration momenta. We need such integrals to calculate the terms of the expansion (19) for case 2. We shall

present the result valid for a more general case, when the integrand is a product of two arbitrary scalar functions. Let

$$K [\text{something}] = \int \int d^n p d^n q \{ \text{something} \} F_1(k^2, p^2, (pk)) F_2(k^2, q^2, (qk)). \quad (63)$$

Then

$$K [(pq)^N] = \frac{N!}{2^N (k^2)^N} \sum_{\{j\}} \frac{j_3!}{j!} \frac{\left(\frac{n}{2}\right)_{j_3}}{\left(\frac{n-2}{2}\right)_{j_3} (n-2)_{j_3} \left(\frac{n}{2}\right)_{j+j_3}} \\ \times K \left[ (k^2)^{N/2} (p^2)^{N/2} C_{j_3}^{(n-2)/2} \left( \frac{(kp)}{\sqrt{k^2 p^2}} \right) (k^2)^{N/2} (q^2)^{N/2} C_{j_3}^{(n-2)/2} \left( \frac{(kq)}{\sqrt{k^2 q^2}} \right) \right], \quad (64)$$

where the sum goes over all  $\{j, j_3\}$  such that  $2j + j_3 = N$  (so, in fact this is a one-fold finite sum) while<sup>17</sup>

$$C_j^\gamma(x) = \sum_{l=0}^{[j/2]} \frac{(-1)^l (\gamma)_{j-l}}{l!(j-2l)!} (2x)^{j-2l} \quad (65)$$

are Gegenbauer polynomials. The result (64) can be derived using the formulae presented in ref. [27]<sup>18</sup>. Using the definition (65), it is easy to see that we do not get neither negative powers nor square roots of  $p^2$  and  $q^2$  on the r.h.s. of eq. (64).

## Appendix B. The $S_1$ term for the case 1 with general masses

Here we present the result for the  $S_1$  contribution to the expansion (19) for the case 1 with different masses<sup>19</sup>:

$$S_1 = -\frac{1}{4(M_1^2 - M_3^2)^3 (M_4^2 - M_3^2)^3 (M_1^2 - M_4^2)^3} \left\{ 2 \left( \tau(m_2, m_5; k^2) + \frac{1}{2} \ln \frac{M_1^2 M_4^2}{m_2^2 m_5^2} + 2 \right) \right. \\ \times \left[ M_1^2 (M_4^2 - M_3^2)^3 \ln \frac{M_1^2}{M_3^2} \left( M_1^2 (M_1^2 - M_3^2) (M_1^2 - M_4^2) (k^2 + m_2^2 - m_5^2) \right. \right. \\ \left. \left. + 2M_4^2 (M_1^2 - M_3^2)^2 k^2 + 2M_3^2 (M_1^2 - M_4^2)^2 m_2^2 \right) \right. \\ \left. + M_4^2 (M_1^2 - M_3^2)^3 \ln \frac{M_4^2}{M_3^2} \left( M_4^2 (M_4^2 - M_3^2) (M_1^2 - M_4^2) (k^2 - m_2^2 + m_5^2) \right. \right. \\ \left. \left. - 2M_1^2 (M_4^2 - M_3^2)^2 k^2 - 2M_3^2 (M_1^2 - M_4^2)^2 m_5^2 \right) \right. \\ \left. - (M_1^2 - M_3^2) (M_4^2 - M_3^2) (M_1^2 - M_4^2) \right. \\ \left. \times \left( (M_1^2 - M_3^2) (M_4^2 - M_3^2) \left( M_1^2 (M_4^2 - M_3^2) + M_4^2 (M_1^2 - M_3^2) \right) k^2 \right. \right. \\ \left. \left. - (M_1^2 - M_3^2) (M_4^2 - M_3^2) (M_1^2 - M_4^2) \right) \right] \right\}$$

<sup>17</sup>The symbol  $[j/2]$  denotes the integer part of  $j/2$ .

<sup>18</sup>It is interesting to note that eq. (64) has a structure similar to eq. (43) of [28], with  $N_1 = N_2 = N$  and  $j_1 = j_2 = j$ .

<sup>19</sup>The lowest term,  $S_0$ , is given by eq. (23).

$$\begin{aligned}
& +(M_4^2 - M_3^2)(M_1^2 - M_4^2) \left( M_1^2(M_4^2 - M_3^2) - M_3^2(M_1^2 - M_4^2) \right) m_2^2 \\
& - (M_1^2 - M_3^2)(M_1^2 - M_4^2) \left( M_3^2(M_1^2 - M_4^2) + M_4^2(M_1^2 - M_3^2) \right) m_5^2 \Big] \\
& - \left( \mathcal{H}(M_1^2, M_3^2) - \frac{1}{2} \ln^2 \frac{M_1^2}{M_3^2} \right) (M_4^2 - M_3^2)^3 \left[ 4M_1^2 M_3^2 (M_1^2 - M_4^2)^2 (m_2^2 + m_5^2) \right. \\
& \quad \left. + (M_1^2 - M_3^2) \left( (M_1^2 + M_4^2)(M_1^4 - M_3^4) + 2M_1^2 M_3^2 (M_1^2 - M_4^2) \right) k^2 \right] \\
& + \left( \mathcal{H}(M_4^2, M_3^2) - \frac{1}{2} \ln^2 \frac{M_4^2}{M_3^2} \right) (M_1^2 - M_3^2)^3 \left[ 4M_4^2 M_3^2 (M_1^2 - M_4^2)^2 (m_2^2 + m_5^2) \right. \\
& \quad \left. + (M_4^2 - M_3^2) \left( (M_1^2 + M_4^2)(M_4^4 - M_3^4) - 2M_4^2 M_3^2 (M_1^2 - M_4^2) \right) k^2 \right] \\
& + 2(M_1^2 - M_3^2)(M_1^2 - M_4^2) m_2^2 \ln \frac{m_2^2}{M_3^2} \\
& \times \left[ (M_4^2 - M_3^2)^3 M_1^4 \ln \frac{M_1^2}{M_3^2} - (M_1^2 - M_3^2)^2 \left( M_4^2(M_4^2 - M_3^2) - 2M_3^2(M_1^2 - M_4^2) \right) M_4^2 \ln \frac{M_4^2}{M_3^2} \right. \\
& \quad \left. - (M_1^2 - M_3^2)(M_4^2 - M_3^2)(M_1^2 - M_4^2) \left( M_1^2(M_4^2 - M_3^2) + 2M_3^2(M_1^2 - M_4^2) \right) \right] \\
& + 2(M_4^2 - M_3^2)(M_1^2 - M_4^2) m_5^2 \ln \frac{m_5^2}{M_3^2} \\
& \times \left[ (M_1^2 - M_3^2)^3 M_4^4 \ln \frac{M_4^2}{M_3^2} - (M_4^2 - M_3^2)^2 \left( M_1^2(M_1^2 - M_3^2) + 2M_3^2(M_1^2 - M_4^2) \right) M_1^2 \ln \frac{M_1^2}{M_3^2} \right. \\
& \quad \left. + (M_1^2 - M_3^2)(M_4^2 - M_3^2)(M_1^2 - M_4^2) \left( M_4^2(M_1^2 - M_3^2) - 2M_3^2(M_1^2 - M_4^2) \right) \right] \\
& - M_1^2(M_4^2 - M_3^2)^3 \ln^2 \frac{M_1^2}{M_3^2} \left[ M_1^2(M_1^2 - M_3^2)(M_1^2 - M_4^2)(k^2 + m_2^2 - m_5^2) \right. \\
& \quad \left. + 2M_4^2(M_1^2 - M_3^2)^2 k^2 + 2M_3^2(M_1^2 - M_4^2)^2 m_2^2 \right] \\
& - M_4^2(M_1^2 - M_3^2)^3 \ln^2 \frac{M_4^2}{M_3^2} \left[ M_4^2(M_4^2 - M_3^2)(M_1^2 - M_4^2)(k^2 - m_2^2 + m_5^2) \right. \\
& \quad \left. - 2M_1^2(M_4^2 - M_3^2)^2 k^2 - 2M_3^2(M_1^2 - M_4^2)^2 m_5^2 \right] \\
& - M_3^2(M_1^2 - M_4^2)^3 \ln \frac{M_1^2}{M_3^2} \ln \frac{M_4^2}{M_3^2} \left[ M_3^2(M_1^2 - M_3^2)(M_4^2 - M_3^2)(k^2 - m_2^2 - m_5^2) \right. \\
& \quad \left. - 2M_1^2(M_4^2 - M_3^2)^2 m_2^2 - 2M_4^2(M_1^2 - M_3^2)^2 m_5^2 \right] \\
& + (M_4^2 - M_3^2)(M_1^2 - M_4^2) \ln \frac{M_1^2}{M_3^2} \left[ M_3^2(M_4^2 - M_3^2)(k^2 - m_2^2 + m_5^2) \right. \\
& \quad \times \left( (M_1^2 - M_4^2) \left( 2M_1^2(M_1^2 - M_4^2) + M_1^4 - M_3^4 \right) - 2(M_1^2 + M_4^2)(M_1^2 - M_3^2)^2 \right) \\
& \quad - (M_4^2 - M_3^2)^2 \left( 8M_1^2 M_3^2 (M_1^2 - M_4^2) + 3M_1^2(M_1^4 - M_3^4) - M_4^2(M_1^2 - M_3^2)^2 \right) m_2^2 \\
& \quad \left. + (M_1^2 - M_3^2) \left( 2M_1^2(M_4^2 - M_3^2)^2(M_4^2 + 2M_3^2) - (M_1^2 - M_3^2)^2(M_1^2 - M_4^2)(M_4^2 + M_3^2) \right) m_5^2 \right] \\
& + (M_1^2 - M_3^2)(M_1^2 - M_4^2) \ln \frac{M_4^2}{M_3^2} \left[ M_3^2(M_1^2 - M_3^2)(k^2 + m_2^2 - m_5^2) \right. \\
& \quad \times \left( (M_1^2 - M_4^2) \left( 2M_4^2(M_1^2 - M_4^2) - M_4^4 + M_3^4 \right) - 2(M_1^2 + M_4^2)(M_4^2 - M_3^2)^2 \right)
\end{aligned}$$

$$\begin{aligned}
& +(M_1^2 - M_3^2)^2 \left( 8M_4^2 M_3^2 (M_1^2 - M_4^2) - 3M_4^2 (M_4^4 - M_3^4) + M_1^2 (M_4^2 - M_3^2)^2 \right) m_5^2 \\
& +(M_4^2 - M_3^2) \left( 2M_4^2 (M_1^2 - M_3^2)^2 (M_1^2 + 2M_3^2) + (M_4^2 - M_3^2)^2 (M_1^2 - M_4^2) (M_1^2 + M_3^2) \right) m_2^2 \Big] \\
& -(M_1^2 - M_3^2) (M_4^2 - M_3^2) (M_1^2 - M_4^2)^2 \Big[ (M_1^2 - M_4^2) (M_1^2 + M_3^2) (M_4^2 + M_3^2) (k^2 - m_2^2 - m_5^2) \\
& \quad - 4M_3^2 (M_1^2 - M_4^2) \left( M_1^2 m_2^2 + M_4^2 m_5^2 + M_3^2 (k^2 - 2m_2^2 - 2m_5^2) \right) \\
& \quad - 2(M_1^2 - M_3^2) (M_4^2 - M_3^2) (M_1^2 m_2^2 - M_4^2 m_5^2) \Big] \Big\} .(66)
\end{aligned}$$

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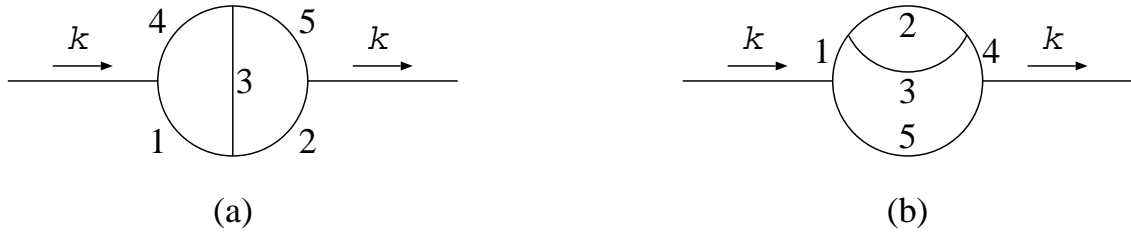


Figure 1: Two-loop self-energy diagrams.

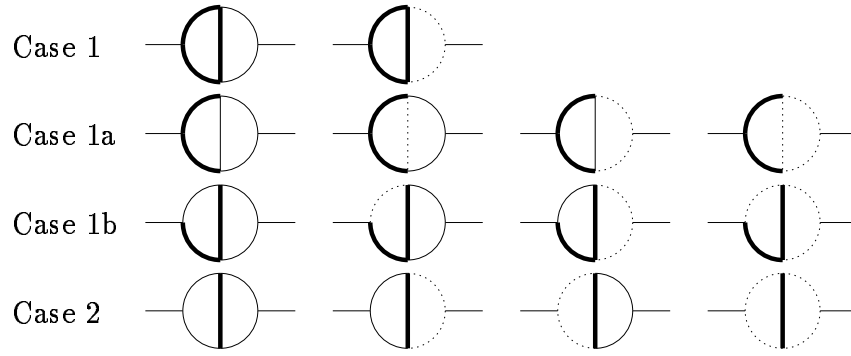


Figure 2: The subgraphs  $\gamma$  to be included in the sum (5).



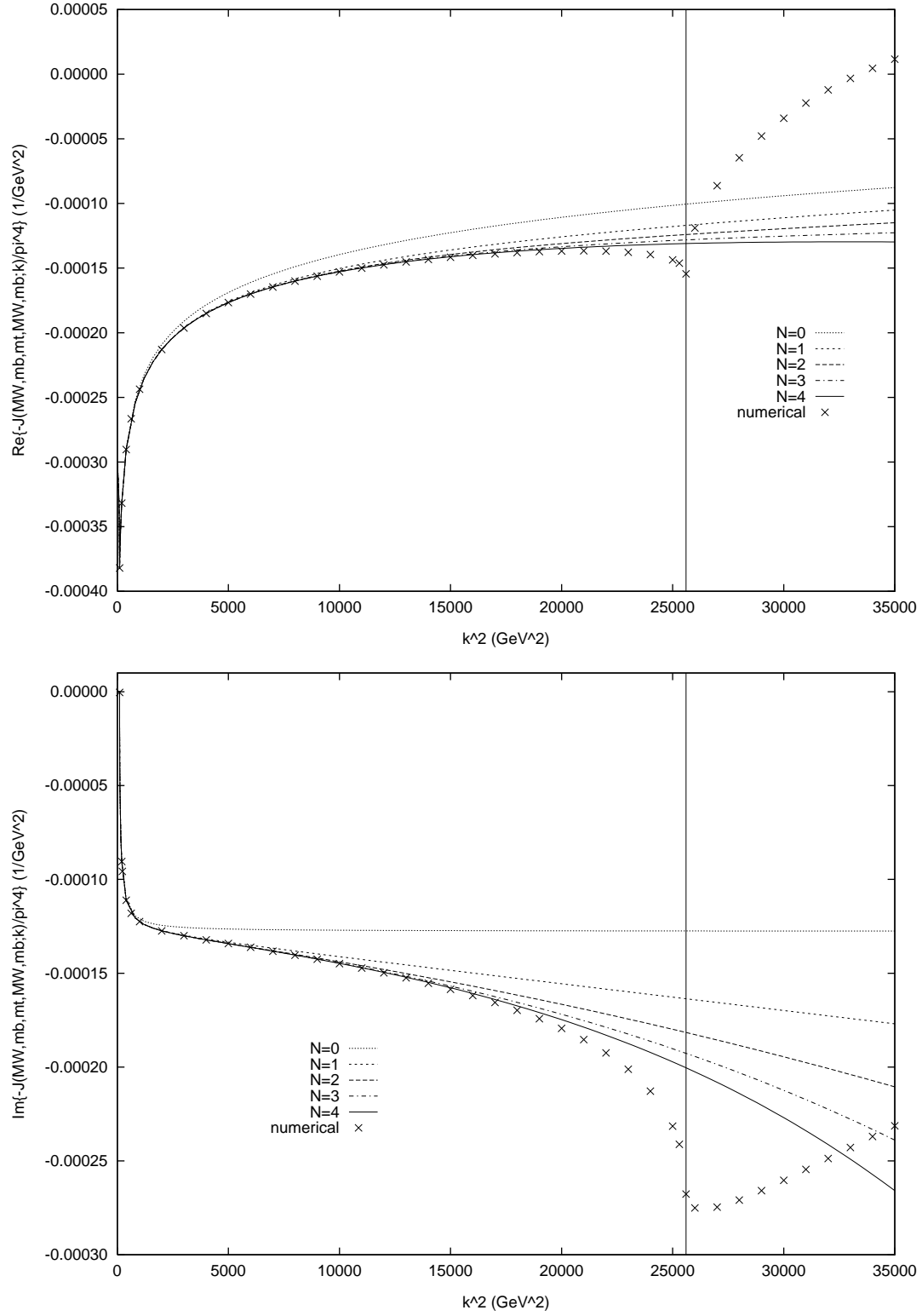


Figure 3: The real and imaginary parts of  $J(M_W, m_b, m_t, M_W, m_b; k)$ .

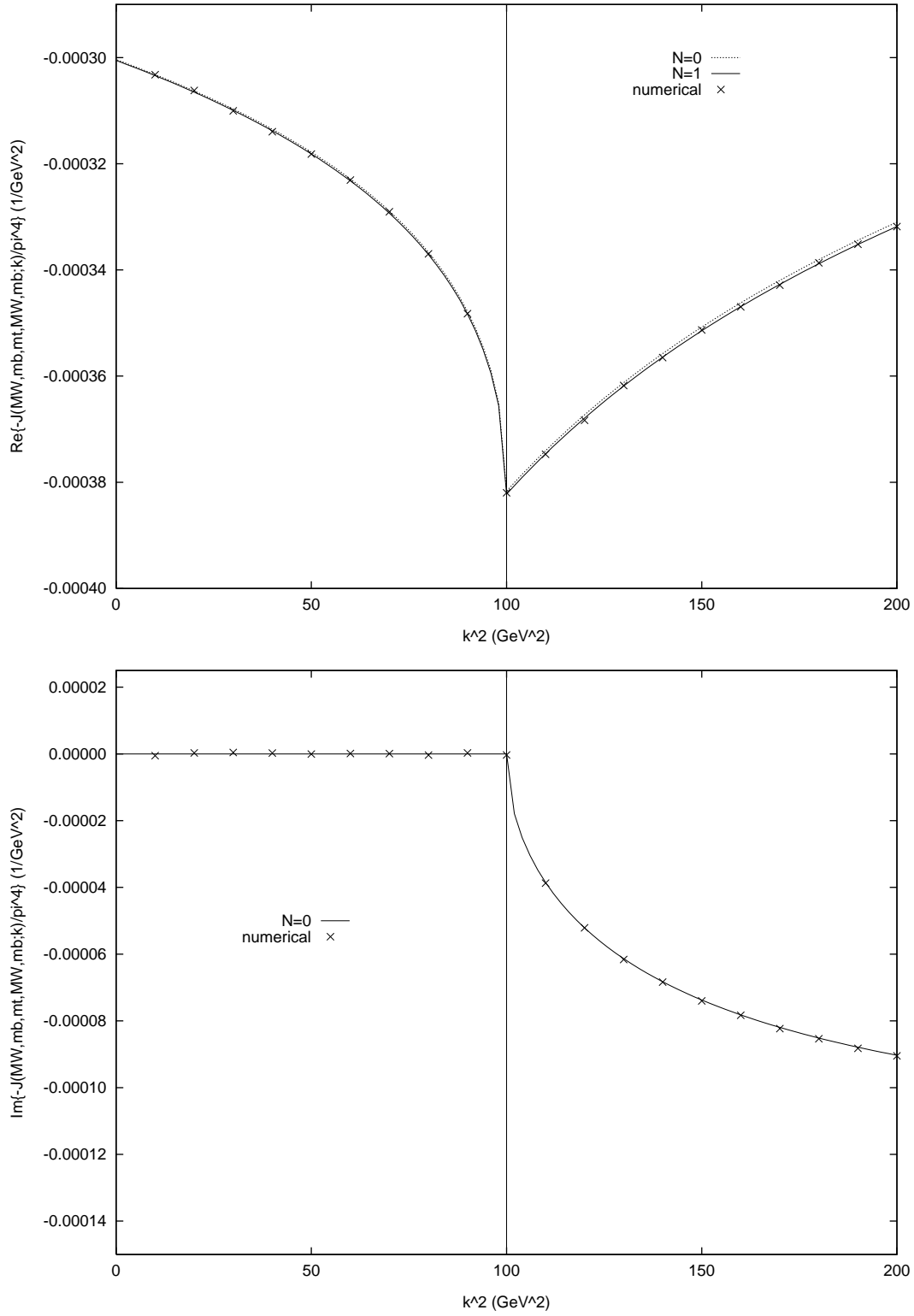


Figure 4: The real and imaginary parts of  $J(M_W, m_b, m_t, M_W, m_b; k)$  at small momenta.

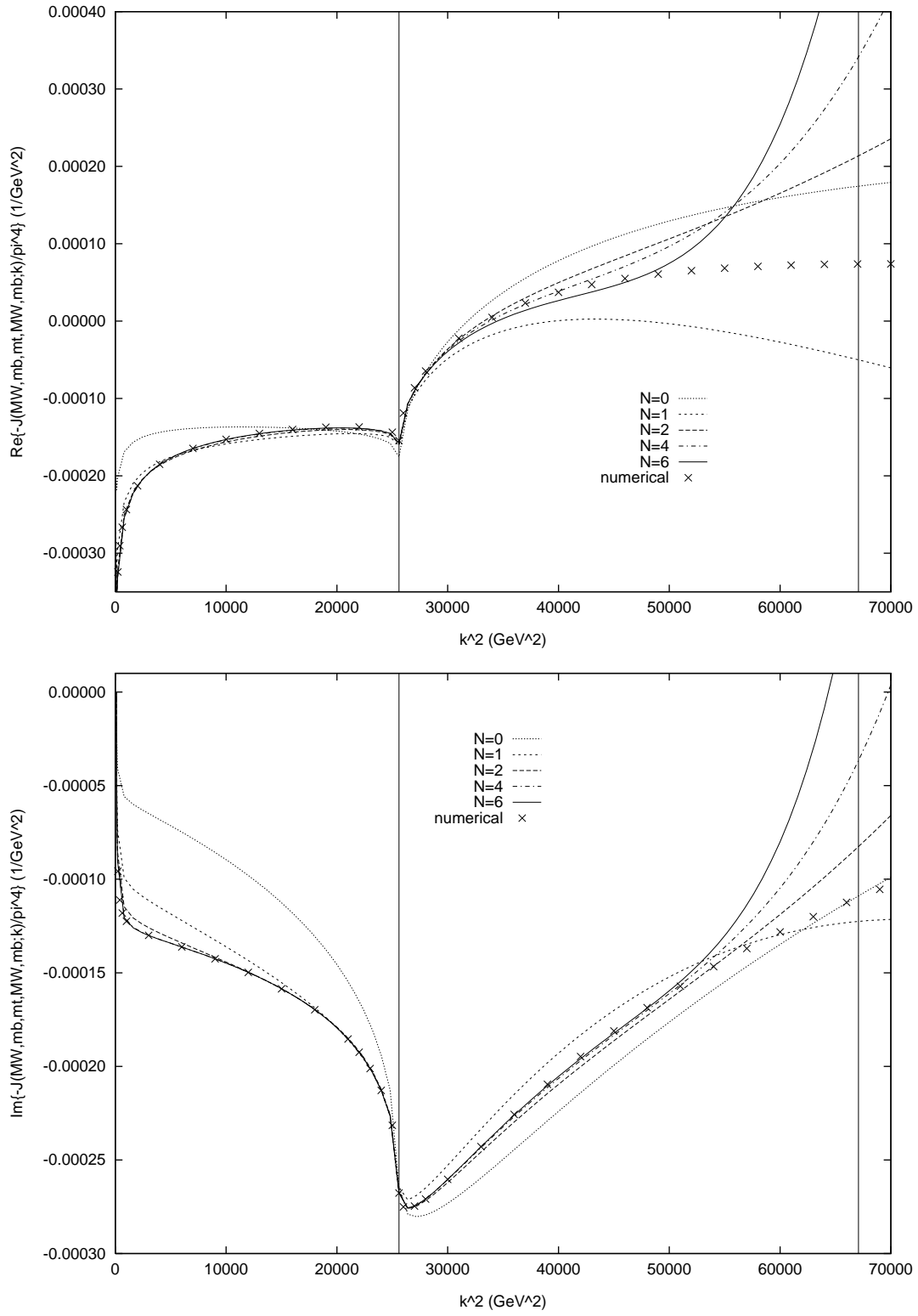


Figure 5: The real and imaginary part of  $J(M_W, m_b, m_t, M_W, m_b; k)$  up to the three-particle threshold.

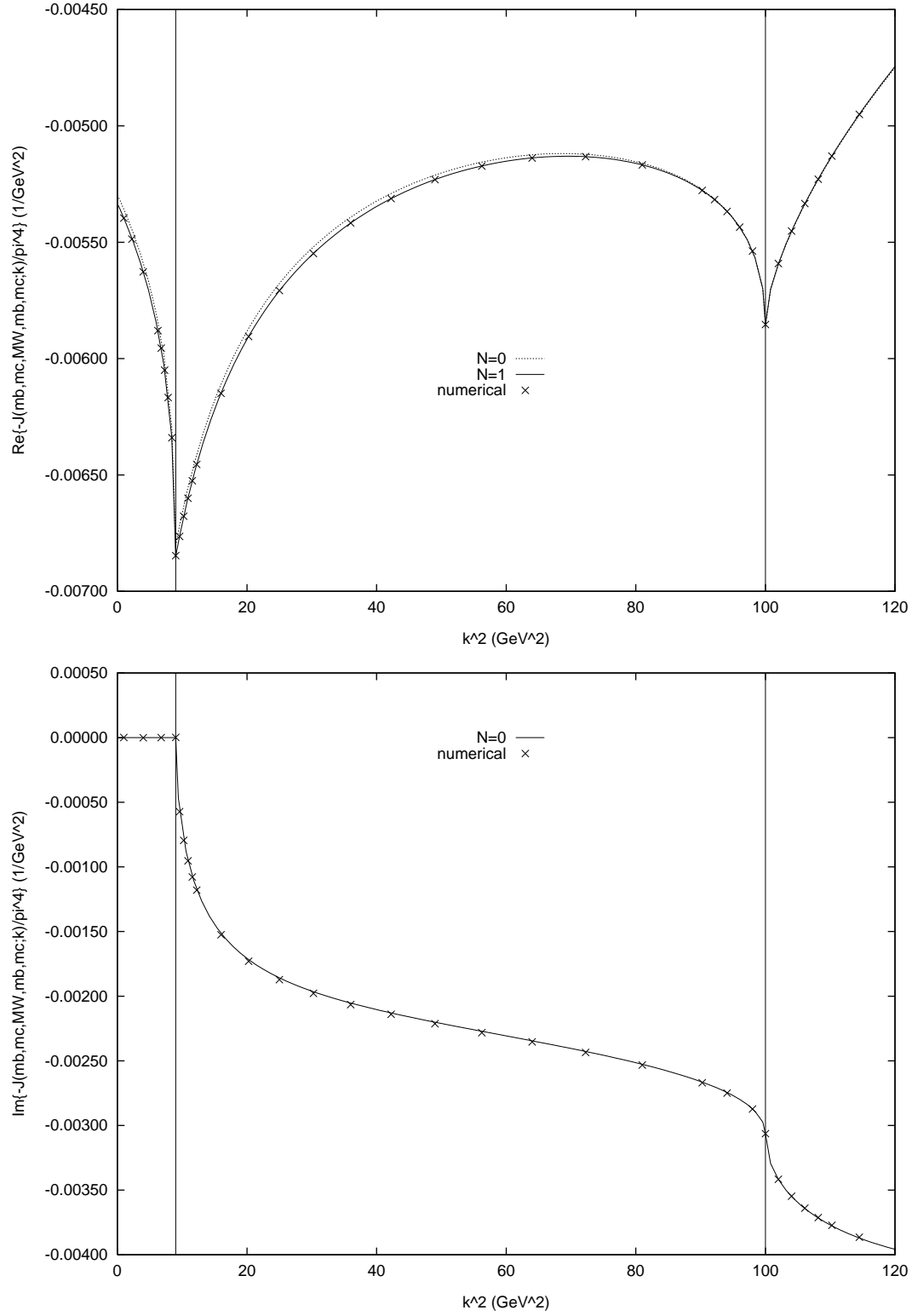


Figure 6: The real and imaginary part of  $J(m_b, m_c, M_W, m_b, m_c; k)$  at small momenta.